Part 1 Online algs

Online algorithms

"Make sequence decisions in the face of uncertainty about the future"

(* There is some cost paid for making a decision.
2 • Can't take choices Back At the end of the sequence of decisions we want to say we didn't spend "much more" than what was best in hindsight

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Def 1: Let OPT be the Alg that makes the best sequence of decisions one could make if we knew the series of requests in advance
A decision-making alg A is r-competitive if

∀ request sequences
$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$

$cost_A(\sigma) \leq r \cdot cost_{opt}(\sigma) + O(1)$

Sk: rental problem

You going on a trip you either rent skis for $1 per day, or buy skis for $B.

However, you don't know how long you will stay

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formal | def

optimal:

if $K \leq B$, rent
else Buy

$Cost_{OPT} = \min\{B, K\}$

Algorithm A:

Rent for the first B days then Buy
$Cost_A = \min\{K, 2B\}$

claim: A is 2-competitive

if $K < B$
$Cost_A = Cost_{OPT}$
$K = K$
if days $\geq B$

$Cost_A = 2 \cdot Cost_{OPT}$
$2B = 2 \cdot B$

Part 2 online Primal/Dual

On line Primal/Dual framework

Recal: LP's

Primal form

minimize $c^T x$

subject to

$Ax \geq b$

$x \geq 0$

Weak duality

$OPT_P \geq OPT_D$

Dual form

max $b^T y$

subject to

$A^T y \leq c$

$y \geq 0$

Online LP's

- instead of seeing A at the beginning we get A 1 row at a time and satisfy each row

- OSS variables throughout, start at 0 and are monotonically non-decreasing

Part 3 Primal Dual Ski rental

Ski-Rental (LP) relaxation

- Costs for k days

Renting costs \$1
Buying costs \$B

$x \in \{0,1\}$ (did we buy?)

$z_i \in \{0,1\}$ (did we rent on day i)

LP: minimize $Bx + \sum_{i=1}^{k} z_i$ (total cost)

subject to:

(If needed to rent/buy) $\rightarrow$ $\forall i \in [1,k]$ $x + z_i \geq 1$

$x \geq 0$

$\forall i$ $z_i \geq 0$

Dual:

$$\max \sum_{i=1}^{k} y_i$$

$$\text{s.t.} \sum_{i=1}^{k} y_i \leq B$$

$\forall i$ $y_i \leq 1$

$y \geq 0$

Algorithm:

- When constraint $x + z_i \geq 1$ arrives

  (1) Do nothing if already satisfied

  (2) otherwise, increase y_i until some dual constraint becomes tight

Set corresponding primal variable to 1

if y_i became 1, set z_i to 1

if $\sum_{i=1}^{t} y_i \leq B$ became 1, set x to 1

- at the end we have a pair of feasible solutions for the primal and the dual

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Proof using lemma

α is 1, β is 2 which implies ALG is 2-competitive

integral vs fractional

Lemma: Approx complementary slackness

Let x, y be feasible solutions to primal / dual

Suppose $\exists\ \alpha, \beta > 0$ s.t.

- $\forall i$, if $x_i > 0$ then $\frac{c_i}{\alpha} \leq \langle (A^T)_i, y \rangle \leq c_i$
- $\forall i$, if $y_i > 0$ then $b_i \leq \langle A_i, x \rangle \leq \beta \cdot b_i$

Then $c^T x \leq \alpha \cdot \beta \cdot b^T y$

Why does this help??

$$b^T y \leq OPT_D \leq OPT_P \leq c^T x \leq \alpha\beta b^T y \leq \alpha\beta\, OPT_P$$

We have a pair of feasible solutions

What if x is bigger than 0 then the constraint is tight, same with z so $\alpha = 1$

if $y_i \leq 0$ then $x + z_i \geq 1$, but we might set both of them to 0 so at worst case we get 2.

$\beta = 2$

Online set cover

Online Set Cover

- Universe of m elements
- Collection of n subsets
  $C = \{S_1, S_2, \ldots, S_n\}$
  $S_i \subseteq [1, \ldots, m]$

Goal: Pick the smallest subset of all S_i that covers the universe

Online: give one element at a time, and see which sets contain that element

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" Set cover is hard to approximate $\log(n)$

Can't do better than $\log(n)$-competitive

Show it is $\log(n) \cdot \log(m)$-competitive

Primal

minimize $\sum_{S \in C} x_S$

s.t. $\forall i \in [m] \quad \sum_{S \ni i} x_S \geq 1$

$x \geq 0$

Dual

maximize $\sum_{i=1}^{m} y_i$

s.t. $\forall S \in C \quad \sum_{i \in S} y_i \leq 1$

$y \geq 0$

Algorithm: (Primal)

- initialize $x = (\frac{1}{n}, \ldots, \frac{1}{n})$
- When we see constraint i
  - Suppose $i \in S_{j_1}, \ldots, S_{j_r}$
  - While constraint is not satisfied run a continuous process where x_S changes according to $\frac{dx_S(t)}{dt} = x_S(t)$
  - run this process simultaneously for all $S_{j_1} \ldots S_{j_r}$ until the constraint is satisfied
  - After x_S evolves T time
  - Whenever we run this evolution, some set in opt is increasing
  - No set in opt evolves for longer than $\log(n)$ seconds
  - Algorithm runs for at most $\log(n)$ seconds

Claim this Algorithm is $\log(n)$-competitive

$$\frac{d(\text{primal obj}(t))}{dt} = \frac{d(x_{S_{j_1}}(t) + \ldots + x_{S_{j_r}}(t))}{dt}$$

$$= x_{S_{j_1}}(t) + \ldots + x_{S_{j_r}}(t) \leq 1$$

$\Rightarrow$ at the end of the alg the primal objective is $\leq \text{opt} \cdot (\ln(m))$

Primal dual view

- Maintain X_S as before
- y initially set to 0
- as we run our evolution we also evolve Y_i

$$\frac{dY_i(t)}{dt} = 1$$

"Primal goes up by at most 1, dual is going up by 1,

Cost of primal is at most the cost of the dual

Since primal is always greater than dual this should be optimal, but it can't be, so why is it not??

- dual is infeasible so we need to scale down
- X is feasible

Claim $\frac{Y}{\ln(n)}$ is dual feasible

Proof Need to show

$$\forall S \in C \quad \sum_{i \in S} Y_i \leq \ln(n)$$

- Whenever $\sum_{i \in C} Y_i$ increases

Then X_S is increasing

- X_S can only increase $\ln(n)$ time steps

$$Cost_D\left(\frac{Y}{\ln(n)}\right) \leq OPT_f \leq cost_P(X)$$

$$\leq Cost_D(Y)$$

$$\Rightarrow \quad cost_P(X) \leq \ln(n) \cdot OPT$$

Making solution Integral

<u>first attempt</u>: At beginning, for each $S \in C$, pick uniform random $\alpha_S \in [0,1]$

- once $x_i \geq \alpha_{S_i}$ take S into our solution

$\Rightarrow E$ cost (integral solution) = cost (fractional solution)

Might not be feasible

Fix: (n is known)

for each S pick

$\alpha_S^{(1)}, \alpha_S^{(2)}, \ldots, \alpha_S^{(r)} \in [0,1]$ uniform at random i.i.d

- $r = C \cdot \ln(n)$
- if $x_S \geq \alpha_S^{(i)}$ for any i then put S into our solution

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Proof of expectation

Show $E(\text{cost}(\text{integral sol})) - E(\text{cost}(\text{fract sol}))$

Pr(a set is included)

$$E(\text{total cost}) = \sum_{S \in C} Pr(X_S \text{ in solution})$$

for each i in fractional solution

$$\text{cost} = \sum x_S$$

each $x_S \in [0,1]$

for a given x_S

$$\sum Pr(x_S \in \text{solution}) = \sum x_S$$

$E(\text{integral cost}) = \text{total fractional cost}$

How much does our "fix" over count?

Min of $\log(n)$ random variables and using that as cutoff

$$Pr(x_S \in \text{Sol}) = Pr(\min < x_S)$$

$$1 - Pr(\min > x_S) = 1 - (1-x)^{\log(n)}$$

$$\leq \log(n) x \quad \text{when } x \in [0,1] \text{ and } k \geq 1$$

$c \approx (1+\frac{1}{B})^B$ Randomized Ski-rental Algorithm $\frac{e}{e+c}\left(1+\frac{1}{(1+\frac{1}{B})^B-1}\right)$

initialize X, all z_i, and $y_i = 0$
Constant c
When we see a constraint

- if $X = 1$, do nothing
- else
  $z_i \leftarrow 1-X$
  $X \leftarrow (1+\frac{1}{B})X + \frac{1}{cB}$
  $y_i \leftarrow 1$

Analysis

$\frac{\text{Primal obj}}{\text{Dual obj}} \quad \left(= 1+\frac{1}{c}\right)$

In every step either

- neither change
- Dual obj increases by 1
  Primal obj increases by
  $= B\cdot\Delta x + z_i$
  $B\cdot\left((1+\frac{1}{B})x + \frac{1}{cB} - x\right)$
  $B\cdot\left(\frac{x}{B} + \frac{1}{cB}\right)$
  $x + \frac{1}{c} + z_i$

$x + \frac{1}{c} + (1-x) = 1+\frac{1}{c}$

Show x and y are feasible

Primal is immediate

Dual feasibility -
Show were in the else B times.

each step we increment x by

$x_1 = \frac{1}{cB}$

x_i are geometric with ratio $r = 1+\frac{1}{B}$

$$\sum_{i=1}^{B} x_i = \frac{1}{cB}\cdot\sum_{j=0}^{B-1} r^j$$

$$= \frac{1}{cB}\cdot\left[\frac{(1+\frac{1}{B})^B - 1}{\frac{1}{B}}\right)$$

$$= \frac{(1+\frac{1}{B})^B - 1}{c}$$

After B days $x = 1$

$c = (1+\frac{1}{B})^B - 1$